

# Extended Hořava Gravity with Physical Ground-State Wavefunction

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## Abstract

We propose a new extended theory of Hořava gravity based on the following three conditions: (i) UV completion, (ii) healthy IR behavior and (iii) a stable vacuum state in quantized version of the theory. Compared with other extended theories, we stress that any realistic theory of gravity must have physical ground states when quantization is performed. To fulfill the three conditions, we softly break the detailed balance but keep its basic structure unchanged. It turns out that the new model constructed in this way can avoid the strong coupling problem and remains power-counting renormalizable, moreover, it has a stable vacuum state by an appropriate choice of parameters.

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# 1 Introduction

Recently a new attempt to formulate a consistent and renormalizable quantum theory of gravity has received extensive attention. This is an ultraviolet(UV) renormalizable theory of gravity proposed by Hořava in [1]. Inspired by the perspectives existed in the theory of dynamical critical systems and quantum criticality, the proposal assumes that the space and time are anisotropic

$$x^i \rightarrow bx^i, \quad t \rightarrow b^z t, \quad (1)$$

where  $z \geq 1$  is the dynamical critical exponent. In the UV regime it has  $z > 1$ . The theory will flow to  $z = 1$  in the infrared (IR) region. The Lorentz invariance is obviously violated as  $z > 1$  but it assumes that there is a foliated diffeomorphism invariance with respect to the spatial sector<sup>2</sup>. By adding higher order spatial derivative terms into the Lagrangian it can reconcile the UV divergence and make the theory renormalizable by power-counting. It is this perspective that enables the proposal to attract a lot of interests in recent literatures. These papers include from the attempts at finding the classical solutions [8] to the application to cosmology[9, 10], and other aspects (see [11] for an incomplete list). In principle, the independent higher order terms which are allowed in the action seems to be extremely large, leading to the theory lack of predictive power. Hořava overcome this difficulty by introducing an additional condition into the theory—the so called “detailed balance”, an idea borrowing from the condensed matter physics.

On the other hand, in Hořava’s original proposal[1], it was argued that the Lorentz invariance can be recovered in the IR limit where  $z$  flows to 1. In this limit the Einstein’s theory naturally appears assuming that a parameter  $\lambda$  (a dimensionless coupling measuring the breaking of the full diffeomorphism group) also flows to 1 in the same limit. However, recent progress indicates that the theory exhibits a pathological behavior at the low energies. Generally speaking, the pathologies include the following two aspects: the strong coupling problem in the IR fixed point[12] and the non-closure of constraint algebra[13, 14]. Essentially, these two pathologies have the same origin. As pointed out in[15], this is mainly due to the fact that the breaking of general covariance by the preferred foliation of space-time introduces a new scalar excitation. A recent effort attempting to overcome these difficulties is an extended theory of the non-projectable Hořava gravity proposed by Blas, Pujolàs and Sibiryakov (BPS)[17]. The key idea of this extended theory (we denote it by BPS theory hereafter) is to improve the IR behavior by breaking the “detailed balance” and introducing a new 3-vector and its higher derivatives into the Lagrangian. As pointed out in [16], this extension could still possess strong coupling at low energies as we consider cubic or higher order Lagrangian, but it is also possible to avoid the strong coupling if higher derivative terms in the action become important below the strong coupling energy scale[17].

So far it seems that the BPS model is an ideal theory of gravity exhibiting healthy behavior at both high and low energies. However, there are at least two obvious obstacles that prevent us from the final theory. First, by giving up the “detailed balance”, the potential term in the action appears to include a large number of terms and hence the number of the parameters

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<sup>2</sup> Although the Lorentz invariance has been verified experimentally at sufficiently large scales, it is possible to have a Lorentz violation at high energies[2, 3, 4, 5]. This possibility also has been partially confirmed in some experiments, see [6] and [7] for examples.

needed in this model would be very large, making the theory lack of predictive power. Second, a well-defined quantized theory of the model constructed in this way cannot be guaranteed in the sense that the model may have unphysical ground states. Therefore, we should refine our model by carefully selecting terms in the action so that the model has a well-defined quantized theory. Meanwhile, to make the theory have predictive power, the number of the parameters in the action should be as less as possible. In this paper, we are paying our attention to these problems and trying to construct our theory of gravity based on the following three conditions: (i) UV completion, in the sense that the candidate theory should be renormalizable in the UV regime; (ii) has a healthy IR behavior, namely, the theory should be free of ghost and does not have strong coupling; (iii) can be well quantized in the sense that the theory has a stable vacuum state (physical ground state).

In performing quantization of our model, we apply the stochastic quantization method[18], which is constructive through stochastic differential equation, so that the question of whether a stable vacuum (ground state) really exists or not can be easily investigated and answered. Also it has the great advantage of no need for gauge-fixing when applied to theories with gauge symmetry. Its equivalence to path integral has been well proved in a lot of literatures (see[19] for example).

The organization of the rest of the paper is as follows. In section 2, we start with a brief review of Hořava gravity and its healthy extension. Section 3 focuses on the power-counting renormalization analysis on our new model. Detailed study on the IR behavior of the model is given in section 4, where we will show that our model is free of the strong coupling problem. In section 5, we pay our attention to the quantization of our theory using stochastic quantization. We will show that the theory has a stable vacuum state if  $\lambda < 1/3$ . Conclusions and discussions are given in the last section.

## 2 Anisotropic theory of gravity

For an anisotropic theory of gravity as suggested by Hořava, a power-counting renormalizable action can be constructed by considering the ADM decomposition of the space-time metric

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \quad (2)$$

where  $N$  and  $N_i$  are the lapse and shift functions respectively. The spatial metric  $g_{ij}$  with  $i, j = 1, 2, 3$  for  $(3+1)$ -dimensional spacetimes has a Euclidean signature. For  $z = 3$  theory, a generic action to be power-counting renormalizable is of the form[1, 15]

$$S = \int dt d^3x \sqrt{g} N \left( \frac{2}{\kappa^2} \mathcal{L}_K - \kappa^2 \mathcal{L}_V \right), \quad (3)$$

where  $g$  denotes the determinant of the spatial metric  $g_{ij}$ . The kinetic term is given by

$$\mathcal{L}_K \equiv \mathcal{O}_K = K_{ij} K^{ij} - \lambda K^2 = K_{ij} G^{ijkl} K_{kl}, \quad (4)$$

where  $K_{ij}$  is defined by

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \quad (5)$$

and  $K \equiv K^i_i$ . The symbol  $G^{ijkl}$  is the generalized De Witt metric

$$G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}, \quad (6)$$

with  $\lambda$  a dimensionless free parameter.

The potential term in (3) which satisfies both the power-counting renormalizable condition and foliation-diffeomorphisms is of the form

$$\mathcal{L}_V = \sum_{n=2}^6 D_n(\Lambda, g_{ij}, R_{ij}, \nabla_i R_{jk}, \dots), \quad (7)$$

where  $D_n(n = 2, \dots, 6)$  denote all possible scalars constructed of  $\Lambda, g_{ij}, R_{ij}, \nabla_i R_{jk}, \dots$  with the same dimension  $n$  and spatial parity. In particular,  $D_2$  is of the form  $-(R - 2\Lambda)$  to have a GR limit. A possible term of  $D_3$  is  $\epsilon^{ijk}\nabla_i R_{jk}$ , but it is excluded by spatial parity.  $D_4$  may include terms like  $R_{ij}R^{ij}$ ,  $\Delta R$  etc.. While the only possible term with spatial parity for  $D_5$  is  $\epsilon^{ijk}R_{il}\nabla_j R_k^l$ . The highest dimension allowed by renormalizable condition is 6 and all terms with dimension 6 constitutes  $D_6$  which has  $R_{ij}R^{jk}R_k^i$ ,  $\nabla_i R_{jk}\nabla^i R^{jk}$  and  $R\Delta R \dots$  as its ingredients.

Recent progress on Hořava gravity turns out, however, that the action constructed as (3) does not have a healthy infrared behavior—it suffers from a strong coupling problem due to the violation of the diffeomorphisms for the full spacetimes. A possible way out of this difficulty was recently suggested in [15] by introducing the potential a set of terms which are constructed from a 3-vector

$$\mathcal{E}_i \equiv \frac{\partial_i N}{N}.$$

Explicitly, the extra terms of potential is

$$\delta\mathcal{L}_V = -\alpha\mathcal{E}_i\mathcal{E}^i + \beta(\mathcal{E}_i\mathcal{E}^i)^2 + \gamma\mathcal{E}_i\Delta\mathcal{E}^i + \delta\mathcal{E}_i\mathcal{E}_j R^{ij} + \dots \quad (8)$$

where  $\alpha, \beta, \gamma, \delta$  are coupling constants and ellipse represents all other possible terms constructed from  $\mathcal{E}_i$  and its covariant derivatives but the following conditions should be satisfied[15]: (a) power-counting renormalizability, this is equivalent to require that all the terms should have dimensions no more than 6, (b) spatial parity and, (c) time-reversal invariance. Action constructed in this way turns out[15] to be renormalizable by power-counting and free of strong coupling problem.

So far it seems that we have a good theory of gravity by constructing the gravity action in the way given above. However, as mentioned in the last section, there are at least two obvious obstacles that prevent us from the final result: (i) the potential term in the action (3) appears to include a large number of terms and hence the number of the parameters needed in this model would be very large, making the theory lack of predictive power, and (ii) a well-defined quantized theory of the model constructed in this way cannot be guaranteed in the sense that the model

may have unphysical ground states (we will show this explicitly in section 5). Motivated by these considerations, we refine our model by carefully selecting terms in the action so as to the model has a well-defined quantized theory. Meanwhile, to make the theory have predictive power, it is better to has parameters as less as possible in the action. Ref. [20] shows that for Hořava gravity it is possible to have a physical ground state, and that the detailed balance structure plays an important role in achieving so. For this reason, we keep the basic structure of Hořava's theory, but add terms that contribute to the IR behavior to softly break it. Explicitly, the action is of the form

$$S = \int d^3x dt \sqrt{g} N \left( \frac{2}{\kappa^2} K_{ij} G^{ijkl} K_{kl} - \frac{\kappa^2}{8} E^{ij} G_{ijkl} E^{kl} + \alpha \mathcal{E}_i \mathcal{E}^i \right), \quad (9)$$

where  $E^{ij}$  is given by

$$\sqrt{g} E^{ij} = \frac{\delta W}{\delta g_{ij}}, \quad (10)$$

with

$$W = \mu_1 \int \omega_3 + \mu_2 \int d^3x \sqrt{g} (R - 2\Lambda_W), \quad (11)$$

where

$$\omega_3 = Tr(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma), \quad (12)$$

and  $\mu_i (i = 1, 2)$  are coupling constant with scaling dimensions  $[\mu_i]_s = i - 1$  and  $[\Lambda_W]_s = 2$ . The model (9) is largely simplified and only very limit parameters are needed. It is also obviously renormalizable by power counting and is free of strong coupling problem since the main contribution of  $\delta \mathcal{L}_V$  in (8) in the IR limit comes from  $\mathcal{E}_i \mathcal{E}^i$ . Meanwhile, the theory (9), when a proper choice of parameters are made, can be well quantized at least in the context of stochastic quantization as will see below. We will give more details in the following sections.

### 3 UV completion

In this section we would like to show, in an explicit way, that the extended theory is power-counting renormalizable. To make the analysis more convenient, one rewrites the action (3) in a more explicit form

$$S = \int d^3x dt \sqrt{g} N \left( \frac{2}{\kappa^2} K_{ij} G^{ijkl} K_{kl} - \sum_{a=2}^6 \lambda_a \mathcal{O}_a \right), \quad (13)$$

where

$$\lambda_6 \equiv \frac{\kappa^2 \mu_1^2}{2}, \quad \lambda_5 \equiv -\frac{\kappa^2 \mu_1 \mu_2}{2}, \quad \lambda_4 \equiv \frac{\kappa^2 \mu_2^2}{8}, \quad \lambda_2 \equiv \frac{\Lambda_W \lambda_4}{3\lambda - 1} \quad (14)$$

and

$$\begin{aligned} \mathcal{O}_2 &= R - 3\Lambda_W - \hat{\alpha} \mathcal{E}_i \mathcal{E}^i, & \mathcal{O}_4 &= R_{ij} R^{ij} - \frac{1 - 4\lambda}{4(1 - 3\lambda)} R^2 \\ \mathcal{O}_5 &= \epsilon^{ijk} R_{il} \nabla_j R_k^l, & \mathcal{O}_6 &= C_{ij} C^{ij}, \end{aligned} \quad (15)$$

where  $\hat{\alpha} \equiv \frac{\alpha}{\lambda_2}$  and  $C_{ij}$  is the Cotton tensor, defined by

$$C^{ij} \equiv \epsilon^{ikl} \nabla_k \left( R^j_l - \frac{1}{4} R \delta^j_l \right). \quad (16)$$

The scaling dimensions of the coefficients of terms in the action (13) are

$$[\kappa^2]_s = z - 3, \quad [\lambda_a]_s = z + 3 - a, \quad [\hat{\alpha}]_s = 0.$$

In the context of Hořava-Lifshitz gravity, the dynamical critical exponent in the UV regime is  $z = 3$ , implying that  $\mathcal{O}_K$  and  $\mathcal{O}_6$  are marginal terms and other terms are relevant. Hence the theory is renormalizable by power counting. While in IR regime, where the dynamical critical exponent is flowed to  $z = 1$ , we find only  $\mathcal{O}_K$  and  $\mathcal{O}_2$  are relevant with  $\mathcal{O}_4$  marginal, in this limit we reach the low-energy effective theory of gravity (up to the  $\mathcal{O}_4$  term).

## 4 IR behavior

To see the IR behavior of the Hořava theory, we investigate the quadratic Lagrangian of (13) by introducing the scalar perturbations of the metric. By adopting the same gauge as the one used in [16], we obtain the scalar perturbations of metric

$$N = e^{\phi(t, \vec{x})}, \quad N_i = \partial_i B(t, \vec{x}), \quad g_{ij} = e^{2\psi(t, \vec{x})} \delta_{ij}. \quad (17)$$

Substituting (17) into the action (13) and integrating by part we obtain the following quadratic terms

$$\mathcal{O}_K^{(2)} = 3(1 - 3\lambda)\dot{\psi}^2 - 2(1 - 3\lambda)\dot{\psi}\Delta B + (1 - \lambda)(\Delta B)^2 \quad (18)$$

$$\mathcal{O}_2^{(2)} = -4\phi\Delta\psi + 2(\partial\psi)^2 - \frac{3}{2}\Lambda_W(\phi + 3\psi)^2 + \hat{\alpha}\phi\Delta\phi \quad (19)$$

$$\mathcal{O}_4^{(2)} = \frac{2(\lambda - 1)}{1 - 3\lambda}\psi\Delta^2\psi, \quad \mathcal{O}_5^{(2)} = \mathcal{O}_6^{(2)} = 0. \quad (20)$$

It is obvious that the above quadratic Lagrangian reduces to those obtained in [16] once we set  $\Lambda_W = 0$ . Following [16] the momentum constraints can be obtained by varying the quadratic action with respect to  $B$ ,

$$\Delta B = \frac{3\lambda - 1}{\lambda - 1}\dot{\psi}. \quad (21)$$

Similarly, varying the quadratic action with respect to  $\phi$  we obtain

$$4\Delta\psi + 3\Lambda_W(\phi + 3\psi) - 2\hat{\alpha}\Delta\phi = 0. \quad (22)$$

To solve the constraint (22) we assume that  $\hat{\alpha} = -2/3$ , then it yields

$$\phi = -3\psi. \quad (23)$$

The action for the extra scalar mode of the theory can be obtained by substituting the constraints (21) and (23) into the quadratic Lagrangian

$$S^{(2)} = - \int d^3x dt \left[ \frac{2}{\kappa^2} \frac{1}{c_\psi^2} \dot{\psi}^2 - \frac{2(\lambda - 1 - 2\Lambda_W)\lambda_4}{3\lambda - 1} (\partial\psi)^2 \right], \quad (24)$$

where  $c_\psi^2 = \frac{1-\lambda}{3\lambda-1}$  is the speed of sound for the mode  $\psi$ . It is straightforward from (24) that the dispersion relation of the propagating mode is

$$\omega^2 = - \left( \kappa^2 c_\psi^2 \frac{2(\lambda - 1 - 2\Lambda_W)\lambda_4}{3\lambda - 1} \right) k^2 \quad (25)$$

From the quadratic action (24) we see that the ghost can be avoided by requiring  $c_\psi^2 < 0$ . This imposes a constraint on  $\lambda$

$$\frac{3\lambda - 1}{\lambda - 1} > 0, \quad (26)$$

implying  $\lambda > 1$  or  $\lambda < 1/3$ . On the other hand, from the dispersion relation (25) the only way to avoid exponential instabilities of the propagating mode  $\psi$  is

$$\frac{\lambda - 1 - 2\Lambda_W}{3\lambda - 1} > 0, \quad (27)$$

assuming  $\lambda_4 = \kappa^2 \mu_2^2 / 8 > 0$ . As (26) is satisfied this can be easily fulfilled by requiring

$$\frac{\Lambda_W}{3\lambda - 1} < 0, \quad (28)$$

which is equivalent to require  $\Lambda_W < 0$  for  $\lambda > 1$  or  $\Lambda_W > 0$  for  $\lambda < 1/3$ .

The above analysis shows that, at least for quadratic action the theory is free of strong coupling problem and exhibits a healthy IR behavior as some conditions are fulfilled.

## 5 Quantization of the theory

Recently most works on Hořava's gravity focus on the IR behavior of the theory and try to refine the model by removing the pathological behavior of the extra mode, as mentioned in the last section. However, there is another most fundamental question should be paid more attention, namely, whether the theory can really be quantized in a consistent and non-perturbative manner? If yes, whether this will put any constraint(s) on the parameters appearing in the action or not? In this section we will, following the work [20], make a detailed analysis of these questions by using the stochastic quantization.

### 5.1 Brief review of stochastic quantization

In this subsection, we give a brief survey of the stochastic quantization. Generally speaking, the stochastic quantization can be performed in the following steps: (1) Transforming the action to

Euclidean version via an analytic continuation to imaginary time; (2) Introducing a fictitious time to the system through which the evolution of fields under random walk can be described. The evolution equation is known as the Langevin equation; (3) Defining the  $n$ -point correlation functions by taking averages over the random noise field with a Gaussian distribution; (4) Identifying the equal time correlators for the field with the corresponding quantum Green's functions as the fictitious time approaches infinity. For stochastic quantization, the key point is that the system is assumed to be equilibrium for large fictitious time. In other words, the Euclidean action is assumed to be bounded from below. The most convenient way to see this point is to investigate the Fokker-Planck equation [21] [22] associated with the equations describing the stochastic dynamic of the system.

As an example, let us consider a free scalar field  $\phi(x)$ . As mentioned we introduce a fictitious time  $\tau$ . Then the Langevin equation, which describes the evolution of the system under random motion, is given by

$$\frac{\partial \phi(x, \tau)}{\partial \tau} = -\frac{\delta S_E}{\delta \phi} + \eta(x, \tau), \quad (29)$$

with  $S_E$  the Euclidean action

$$S_E[\phi] = \int d^d x \left( \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m_0^2 \phi^2(x) \right). \quad (30)$$

The white Gaussian noise  $\eta$  in (29) satisfies

$$\langle \eta(x, \tau) \rangle = 0 \quad \langle \eta(x_1, \tau_1) \eta(x_2, \tau_2) \rangle = 2 \delta(\tau_1 - \tau_2) \delta^d(x_1 - x_2), \quad (31)$$

The  $n$ -point correlation function is define as

$$\langle \phi_\eta(x_1, \tau_1) \dots \phi_\eta(x_k, \tau_k) \rangle = \frac{\int \mathcal{D}[\eta] \phi_\eta(x_1, \tau_1) \dots \phi_\eta(x_k, \tau_k) \exp \left[ -\frac{1}{4} \int d^d x \int d\tau \eta^2(\tau, x) \right]}{\int \mathcal{D}[\eta] \exp \left[ -\frac{1}{4} \int d^d x \int d\tau \eta^2(\tau, x) \right]}. \quad (32)$$

Identifying this correlation function with the corresponding quantum Green's functions as the fictitious time approaches infinity, i.e.,

$$\lim_{\tau \rightarrow \infty} \langle \phi_\eta(x_1, \tau_1) \dots \phi_\eta(x_k, \tau_k) \rangle |_{\tau_1 = \dots = \tau_k = \tau} = \langle \phi_\eta(x_1) \dots \phi_\eta(x_k) \rangle, \quad (33)$$

In particular, for the action given by (30), it is easy to show that the equal time two-point correlation function in phase space is given by

$$\langle \phi(\tau, k) \phi(\tau, k') \rangle = (2\pi)^d \delta^d(k + k') \frac{1}{(k^2 + m_0^2)} \left( 1 - \exp(-2\tau(k^2 + m_0^2)) \right). \quad (34)$$

Therefore, the Euclidean two-point function is recovered as  $\tau \rightarrow \infty$ .

On the other hand, the existence of an equilibrium state can be proved or disproved by studying the corresponding Fokker-Planck equation associated with the Langevin equation. This is given by

$$\frac{\partial P(\phi, \tau)}{\partial \tau} = \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} + \frac{\delta S_E}{\delta \phi} \right) P(\phi, \tau), \quad (35)$$



where  $P$  is the probability density which satisfies the normalization condition

$$\int d\phi P(\phi, \tau) = 1. \quad (36)$$

Solving the Fokker-Planck equation (35) for given  $S_E$  one can obtain the probability density. An equilibrium state of a system is supposed to have a positive and finite  $P$ .

## 5.2 Quantization of BPS model

Although the extended Hořava gravity [15] succeeds in avoiding the strong problem of Hořava's original scheme, it is not guaranteed that the theory can be quantized in a consistent way and that it has a well-defined physical ground state. In this subsection, we would point out that the BPS model in its original form may have unphysical ground states since the candidate ground-state function is not always normalizable.

We start with the BPS action

$$S_{BPS} = \int d^3x dt \left[ \frac{2}{\kappa^2} \mathcal{L}_K - \kappa^2 (\mathcal{L}_V + \delta \mathcal{L}_V) \right] \quad (37)$$

where  $\mathcal{L}_K$ ,  $\mathcal{L}_V$  and  $\delta \mathcal{L}_V$  are given, respectively, by (4), (7) and (8). Performing a wick rotation  $t \rightarrow i\tau$  we obtain the Euclidean action of (37), which is denoted by  $S_E^{bps}$  hereafter. Then the Langevin equation of the BPS theory is [20]

$$\begin{cases} \dot{N} = -\frac{1}{\sqrt{g}} \frac{\delta S_E^{bps}}{\delta N} + \eta, \\ \dot{N}_i = -\frac{1}{\sqrt{g}} \frac{\delta S_E^{bps}}{\delta N^i} + \zeta_a e^a_i, \\ \dot{\mathcal{E}}_i = -\frac{1}{\sqrt{g}} \frac{\delta S_E^{bps}}{\delta \mathcal{E}^i} + \sigma_a e^a_i, \\ \dot{g}^I = -\mathcal{G}^{IJ} \partial_J S_E^{bps} + \xi^A E_A^I, \end{cases} \quad (38)$$

where the dot represents derivative with respect to the fictitious time  $\tau$  and following notations have been introduced:

$$g_{ij} \equiv g^I, \quad \mathcal{G}^{IJ} \equiv \mathcal{G}_{ijkl}, \quad \partial_I S_E^{bps} \equiv \frac{1}{\sqrt{g}} \frac{\delta S_E^{bps}}{\delta g_{ij}}.$$

In Eq. (38), we also have introduced vielbein

$$e_a^i e_b^j g_{ij} = \delta_{ab}, \quad E_A^I E_B^J \mathcal{G}_{IJ} = \delta_{AB}, \quad (39)$$

$$e_a^i e_b^j \delta^{ab} = g^{ij}, \quad E_A^I E_B^J \delta^{AB} = \mathcal{G}^{IJ}. \quad (40)$$

so that noises  $\eta$ ,  $\zeta_a$ ,  $\sigma_a$  and  $\xi^A$  are Gaussian and the following relations hold [20]

$$\langle \eta(x, \tau) \rangle = 0, \quad \langle \zeta^a(x, \tau) \rangle = 0, \quad \langle \sigma^a(x, \tau) \rangle = 0, \quad \langle \xi^A(x, \tau) \rangle = 0, \quad (41)$$

$$\langle \eta(x, \tau) \eta(y, \tau') \rangle = 2\delta(x - y) \delta(\tau - \tau'), \quad (42)$$

$$\langle \zeta^a(x, \tau) \zeta^b(y, \tau') \rangle = 2\delta^{ab} \delta(x - y) \delta(\tau - \tau'), \quad (43)$$

$$\langle \sigma^a(x, \tau) \sigma^b(y, \tau') \rangle = 2\delta^{ab} \delta(x - y) \delta(\tau - \tau'), \quad (44)$$

$$\langle \xi^A(x, \tau) \xi^B(y, \tau') \rangle = 2\delta^{AB} \delta(x - y) \delta(\tau - \tau'). \quad (45)$$

(Here  $x$  stands for Euclidean coordinates  $(x^i, \tau)$ .) The correlation functional then can be defined with respect to  $\eta$ ,  $\zeta^a$ ,  $\sigma_a$  and  $\xi^A$  by

$$\begin{aligned} \langle \mathcal{F}(N, N^i, \mathcal{E}_i, g_I) \rangle &\sim \int \mathcal{D}[\eta] \mathcal{D}[\zeta] \mathcal{D}[\sigma] \mathcal{D}[\xi] \mathcal{F}(N, N^i, \mathcal{E}_i, g_I) \\ &\cdot \exp \left[ -\frac{1}{4} \int d\tau d^3x d\tau \sqrt{g} N (\eta^2 + \zeta^a \zeta_a + \sigma^a \sigma_a + \xi^A \xi_A) \right], \end{aligned} \quad (46)$$

which is obviously Gaussian as desired.

As mentioned in the last subsection, a convenient way to study whether the Langevin process (38) really converges to a stationary equilibrium distribution is to explore the associated Fokker-Planck equation,

$$\frac{\partial Q(N, N^i, \mathcal{E}_i, g_I, \tau)}{\partial \tau} = -\mathcal{H}_{FP} Q(N, N^i, \mathcal{E}_i, g_I, \tau). \quad (47)$$

Here we have introduced a new function  $Q$  which is associated the probability density through

$$Q(N, N^i, \mathcal{E}_i, g_I, \tau) \equiv P(N, N^i, \mathcal{E}_i, g_I, \tau) e^{S_E/2}, \quad (48)$$

where the probability density functional is given by

$$P(N, N^i, \mathcal{E}_i, g_I, \tau) = \frac{\exp \left[ -\frac{1}{4} \int d\tau d^3x d\tau \sqrt{g} N (\eta^2 + \zeta^a \zeta_a + \sigma^a \sigma_a + \xi^A \xi_A) \right]}{\int \mathcal{D}[\eta] \mathcal{D}[\zeta] \mathcal{D}[\sigma] \mathcal{D}[\xi] \exp \left[ -\frac{1}{4} \int d\tau d^3x d\tau \sqrt{g} N (\eta^2 + \zeta^a \zeta_a + \sigma^a \sigma_a + \xi^A \xi_A) \right]}. \quad (49)$$

The Fokker-Planck Hamiltonian  $\mathcal{H}_{FP}$  in (47) is of the form

$$\mathcal{H}_{FP} = a^\dagger a + g^{ij} a_i^\dagger a_j + g^{ij} \tilde{a}_i^\dagger \tilde{a}_j + \mathcal{G}^{IJ} \mathcal{A}_I^\dagger \mathcal{A}_J. \quad (50)$$

Here

$$a = i\pi + \frac{1}{2} \frac{1}{\sqrt{g}} \frac{\delta S_E^{bps}}{\delta N}, \quad a^i = i\pi^i + \frac{1}{2} \frac{1}{\sqrt{g}} \frac{\delta S_E^{bps}}{\delta N_i}, \quad \tilde{a}^i = i\tilde{\pi}^i + \frac{1}{2} \frac{1}{\sqrt{g}} \frac{\delta S_E^{bps}}{\delta \mathcal{E}_i}, \quad \mathcal{A}^I = i\pi^I + \frac{1}{2} \partial^I S_E^{bps},$$

with  $\pi$ ,  $\pi^i$ ,  $\tilde{\pi}^i$  and  $\pi^I$ , respectively, the conjugate momenta of  $N$ ,  $N^i$ ,  $\mathcal{E}_i$  and  $g^I$ :  $\pi = -i \frac{1}{\sqrt{g}} \frac{\delta}{\delta N}$ ,  $\pi^i = -i \frac{1}{\sqrt{g}} \frac{\delta}{\delta N_i}$ ,  $\tilde{\pi}^i = -i \frac{1}{\sqrt{g}} \frac{\delta}{\delta \mathcal{E}_i}$ ,  $\pi_I = -i \partial_I$ . The time independent eigenvalue equation associated with Eq. (47) is

$$\mathcal{H}_{FP} Q_n(N, N^i, \mathcal{E}_i, g_I, \tau) = E_n Q_n(N, N^i, \mathcal{E}_i, g_I, \tau). \quad (51)$$

The solutions of Eq. (47) lead to the probability density

$$P(N, N^i, \mathcal{E}_i, g_I, \tau) = \sum_{n=0}^{\infty} a_n Q_n(N, N^i, \mathcal{E}_i, g_I) e^{-S_E^{bps}/2 - E_n \tau}. \quad (52)$$

From (52) we show that the theory will approach an equilibrium state  $Q_0(N, N^i, \mathcal{E}_i, g_I) = e^{-S_E^{bps}/2}$  for large  $\tau$  if and only if all  $E_n > 0$  ( $n > 0$  and with  $E_0 = 0$ ). This is equivalent to find the

condition(s) under which the Fokker-Planck Hamiltonian (50) is non-negative definite. Following the analysis made in [20] we show that this can be fulfilled by requiring a positive definite De Witt metric  $\mathcal{G}^{IJ}$ , i.e.,  $\lambda < 1/3$ . The theory then approaches an equilibrium

$$P_0(N, N^i, \mathcal{E}_i, g_I) \equiv \lim_{\tau \rightarrow \infty} P(N, N^i, \mathcal{E}_i, g_I, \tau) = a_0 e^{-S_E^{bps}}, \quad (53)$$

where

$$a_0 = \frac{1}{\int \mathcal{D}[N] \mathcal{D}[N_i] \mathcal{D}[\mathcal{E}_i] \mathcal{D}[g_I] e^{-S_E^{bps}(N, N^i, \mathcal{E}_i, g_I)}}, \quad (54)$$

is the normalization constant. Note that the stationary candidate of equilibrium state  $P_0$  in (53) is far from a genuine physical ground state. In other words, the normalization constant  $a_0$  in (54) is not guaranteed to be finite. It follows from (54) that the normalizable ground state is achieved by requiring a positive definite Euclidean action  $S_E^{bps}$ . While from (37) we see the action  $S_E^{bps}$  is not always positive definite, implying that some unphysical ground states appear. To cure this problem more constraints have to be imposed on the potential terms.

### 5.3 Stochastic quantization of our model

In this subsection we would like to propose a possible prescription for removing the unphysical vacuum state. Inspired by the result of [20], we found a possible way out is to keep the basic structure of “detailed balance”. However, there are a lot of literatures (see [12] for example) show that the strict “detailed balance” will lead to a catastrophe of the theory—the strong coupling problem as mentioned in the previous part of the paper. To avoid the strong coupling, we have to violate the detailed balance structure. To coordinate these two apparently incompatible conditions smoothly, on one hand, we softly break the detailed balance, on the other hand, we keep the basic structure of the detailed balance. This leads to our extended action (9) of Hořava gravity. This action violates the detailed balance by introducing an extra term  $\mathcal{E}_i \mathcal{E}^i$  whose presence cures the strong coupling problem as analysed in Sec.4. Meantime, it keeps the basic structure of detailed balance which leads to a cure of the unphysical ground states as will see below.

To see this explicitly, we write down the Euclidean action of our model (9),

$$S_E = \int d^3x d\tau \sqrt{g} N \left( \frac{2}{\kappa^2} K_{ij} G^{ijkl} K_{kl} - \frac{\kappa^2}{8} E^{ij} G_{ijkl} E^{kl} + \alpha \mathcal{E}_i \mathcal{E}^i \right), \quad (55)$$

Repeating the procedures given in the last subsection we can quantize the theory using the stochastic quantization, and, similar to the case of BPS theory, we obtain the following solution of the Fokker-Planck equation

$$P(N, N^i, \mathcal{E}_i, g_I, \tau) = \sum_{n=0}^{\infty} a_n Q_n(N, N^i, \mathcal{E}_i, g_I) e^{-S_E/2 - E_n \tau}, \quad (56)$$

where  $S_E$  is given by (55). Therefore, the theory will approach an equilibrium state  $Q_0(N, N^i, \mathcal{E}_i, g_I) = e^{-S_E/2}$  for large  $\tau$  as long as the De Witt metric  $\mathcal{G}^{IJ}$  is positive definite, or equivalently,  $\lambda < 1/3$ .

The candidate equilibrium state of the theory is

$$P_0(N, N^i, \mathcal{E}_i, g_I) \equiv \lim_{\tau \rightarrow \infty} P(N, N^i, \mathcal{E}_i, g_I, \tau) = a_0 e^{-S_E}, \quad (57)$$

where again

$$a_0 = \frac{1}{\int \mathcal{D}[N] \mathcal{D}[N_i] \mathcal{D}[\mathcal{E}_i] \mathcal{D}[g_I] e^{-S_E(N, N^i, \mathcal{E}_i, g_I)}}, \quad (58)$$

is the normalization constant. As mentioned in the last subsection, the key to obtain a stable vacuum state (or physical ground state) is that the Euclidean action in (58) must be positive definite. In our model this can be achieved by requiring that both the De Witt metric and  $\alpha$  are positive definite. Explicitly, we rewrite the action (55) as

$$S_E = \int d^3x d\tau \sqrt{g} N \left[ \frac{2}{\kappa^2} \mathcal{G}^{IJ} (K_I K_J - \frac{\kappa^4}{16} E_I E_J) + \alpha g^{ij} \mathcal{E}_i \mathcal{E}_j \right], \quad (59)$$

where  $E_I = \partial_I W$  with  $W$  given by (11). Therefore,  $S_E$  is positive definite for  $\lambda < 1/3$  and  $\alpha > 0$ . In Sec. 4 we have chosen  $\alpha = -\frac{2}{3}\lambda_2$  with  $\lambda_2$  is defined in (14). It is straightforward to show that the condition to have  $\alpha > 0$  is

$$\frac{\Lambda_W}{3\lambda - 1} < 0.$$

This is precisely the condition (28) with which the theory is free of the strong coupling problem. This condition is equivalent to require  $\lambda < 1/3$  for  $\Lambda_W > 0$ . As a consequence, the state (57) is indeed a physical ground state if  $\lambda < 1/3$ .

## 6 Conclusions and discussions

Based on three conditions: (i) UV completion, (ii) healthy IR behavior and (iii) a stable vacuum state, we have constructed a new extension of the Hořava's gravity. In some sense, this model is an improvement of the BPS model by imposing an extra constraint—the condition with which the theory has a stable vacuum—on the theory. This is achieved by keeping the basic “detailed balance” structure but adding the terms curing the IR pathologies in the action. There are at least three merits when construct theories in this way: First, it puts strong constraints on the number of the allowed terms in the action, hence makes the theory has predictive power; Second, it makes the Euclidean action of the theory bounded from below when  $\lambda < 1/3$  is fulfilled. This is a key condition to have a stable vacuum state for theories when we are performing stochastic quantization or path integral quantization. Third, it provides a possible way in avoiding the strong coupling problem at low energies. Indeed, our analyses made in this paper show that the theory constructed in this way can fulfill all the three conditions mentioned above assuming the parameter  $\lambda$  satisfies some conditions in different energy scales.

One point deserves further investigation is to check whether our model can really avoid the strong coupling problem when we are expanding the Lagrangian to higher order. Although the present paper show that the theory exhibits a healthy IR behavior for the quadratic Lagrangian,

this is not guaranteed for higher order Lagrangian. This is equivalent to check if there is a new scale other than the Planck scale for suppressing the higher derivative terms so that these terms become important before the strong coupling appears [17]. Meanwhile, it is worthy of further study on the Hamiltonian formalism of our model so as to find the constraint structure of the theory.

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